Probabilistic Methods in Combinatorics

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Solutions to Assignment 3

Problem 1. Let H be a graph and let $n > |V(H)|$ be an integer. Suppose that there is a graph on n vertices and m edges that does not contain a copy of H, and let $k > \frac{n^2 \log n}{m}$. Show that the edges of K_n can be coloured with k colours such that there is no monochromatic copy of H.

Solution. Let G be a graph with n vertices and m edges which does not contain H as a subgraph. Let $G_1, \dots, G_k$ random copies of G inside K_n. That is, we define uniformly random bijections $f_1, \dots, f_k : V(G) \to V(K_n)$ and let G_i be $f_i(G)$.

For any edge $e \in E(K_n)$, let the colour of e be the smallest i such that $e \in G_i$. If such i does not exist, let us call the colouring a failure.

Now let us estimate the probability that the colouring is a failure. The colouring fails at e if no subgraph G_i contains e. Since each G_i contains m edges out of the $\binom{n}{2}$ edges of K_n, the probability that a fixed G_i does not contain e is $1 - \frac{m}{\binom{n}{2}}$. Since the various G_i's are independent, the probability that no G_i contains e is $\left(1 - \frac{m}{\binom{n}{2}}\right)^k \leq \exp(-\frac{mk}{\binom{n}{2}}) \leq \exp(-\frac{2mk}{n^2}) < \exp(-2\log n) = n^{-2}$. Hence, the probability that there exists an edge $e \in E(K_n)$ which is not contained in any G_i is less than 1, so with positive probability we get a colouring of K_n.

Now each colour class is a subgraph of some G_i, so it is a subgraph of G, therefore no colour class contains a copy of H.

Problem 2. Show that there is a positive constant $c > 0$ such that for any positive integer n there exists a graph $G = (V, E)$ such that

- $|V| = n$,
- $|E| \geq cn^{8/7}$,
- G does not contain C_8 as a subgraph.

Solution. Let H be a random graph $G(n, p)$. We take $p = \alpha n^{-6/7}$. Denote by X the number of copies of C_8 in H. Then

$$\mathbb{E}[X] \le n^8 p^8 = (\alpha n^{1/7})^8 = \alpha^8 n^{8/7},$$

where the inequality follows as there are at most n^8 potential copies of C_8 (n ways to pick the first vertex in the graph, n ways for the second vertex, etc.), and each potential copy is in H with probability p^8 (because C_8 has eight edges). Also

$$\mathbb{E}[e(H)] = \binom{n}{2} p \ge \frac{n^2}{3} \cdot \alpha n^{-6/7} = \frac{\alpha}{3} n^{8/7}.$$

We form a graph H' by removing one edge from each copy of C_8. We thus remove at most X edges from H, so the number of edges in H' satisfies

$$\mathbb{E}[e(H')] \ge \mathbb{E}[e(H) - X] = \mathbb{E}[e(H)] - \mathbb{E}[X] \ge (\frac{\alpha}{3} - \alpha^8) n^{8/7}.$$

Take, say, $\alpha = 1/2$ and pick an instance of H such that $e(H') \ge (\frac{\alpha}{3} - \alpha^8) n^{8/7} \ge (1/12) n^{8/7}$. Then H' satisfies the above requirements with $c = 1/12$.

Problem 3. A collection $\mathcal{F}$ of subsets of $[n]$ is called *k-independent* if for every k distinct sets $F_1, \dots, F_k \in \mathcal{F}$, all of the 2^k intersections $\bigcap_{i=1}^k G_i$ are non-empty, where each G_i is either F_i or its complement $[n] \setminus F_i$. Prove that for $k \ge 6$ there is a k-independent family of subsets of $[n]$ of size at least $\left\lfloor e^{n/(k2^k)} \right\rfloor$ (exponentially large!).

Solution. Let $m = \left\lfloor e^{n/(k2^k)} \right\rfloor$. Let $S_1, S_2, \dots, S_m$ be independent, uniformly random subsets of $[n]$. Let us estimate the probability that $S_1, \dots, S_k$ are k-independent. The probability that $\bigcap_{i=1}^k S_i$ is empty is $(1 - \frac{1}{2^k})^n$ since for each $1 \le t \le n$, the probability that $t \notin \bigcap_{i=1}^k S_i$ is $1 - \frac{1}{2^k}$.

Since S_i and $[n] \setminus S_i$ have the same distribution, we also have $\mathbb{P}(\bigcap_{i=1}^k G_i = \emptyset) = (1 - \frac{1}{2^k})^n$ whenever each G_i is either F_i or $[n] \setminus F_i$. Hence, by the union bound, the probability that $S_1, \dots, S_k$ are not k-independent is at most $2^k (1 - \frac{1}{2^k})^n \le 2^k e^{-n/2^k}$.

Similarly, for any $1 \le i_1 < i_2 < \cdots < i_k \le m$, the probability that $S_{i_1}, S_{i_2}, \dots, S_{i_k}$ are not k-independent is at most $2^k e^{-n/2^k}$. The set $\{S_1, \dots, S_m\}$ is not k-independent if some k-subset of it is not k-independent. By the union bound, this has probability at most $\binom{m}{k} \cdot 2^k e^{-n/2^k} \le \frac{m^k}{k!} 2^k e^{-n/2^k} < m^k e^{-n/2^k} = (m e^{-n/(k2^k)})^k \le 1$. So with positive probability $\{S_1, \dots, S_m\}$ defines a k-independent family of size $\left\lfloor e^{n/(k2^k)} \right\rfloor$.

Problem 4. Let $G = (V, E)$ be a graph on n vertices, with minimum degree $\delta > 1$. We say that a set $U \subseteq V$ is dominating if every vertex $v \in V \setminus U$ has at least one neighbour in U.

Show that G has a dominating set of size at most $\frac{\log(\delta+1)+1}{\delta+1}n$.

Solution. Let $0 < p < 1$, to be defined later. Let A be a set of vertices, chosen randomly by putting every vertex of G in A with probability p, independently. Let B be the (random) set of vertices that are not in A and that do not have a neighbour in A. Note that $A \cup B$ is a dominating set. Moreover,

$$\begin{aligned}
\mathbb{E}[|A|] &= np \\
\mathbb{E}[|B|] &= \sum_{u \in V(G)} (1-p)^{d(u)+1} \le n(1-p)^{\delta+1}.
\end{aligned}$$

(The first equality in the expectation of B follows because u being in B means that u and all of its neighbours are not in A. The next inequality follows from the minimum degree condition.) Put $p = \frac{\log(\delta+1)}{\delta+1}$. Then

$$\begin{aligned}
\mathbb{E}[|A \cup B|] &\le n\left(p + (1-p)^{\delta+1}\right) \le n\left(p + e^{-p(\delta+1)}\right) \\
&= n\left(\frac{\log(\delta+1)}{\delta+1} + e^{-\log(\delta+1)}\right) = \frac{1+\log(\delta+1)}{\delta+1} \cdot n.
\end{aligned}$$